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Abstract
This article reconsiders the theory of existence of efficient allocations and equilibria when consumption sets are unbounded below under the assumption that agents have incomplete preferences. It is motivated by an example in the theory of assets with short-selling where there is risk and ambiguity. Agents have Bewley’s incomplete preferences. As an inertia principle is assumed in markets, equilibria are individually rational. It is shown that a necessary and sufficient for existence of an individually rational efficient allocation or of an equilibrium is that the relative interiors of the risk adjusted sets of probabilities intersect. The more risk averse, the more ambiguity averse the agents, the more likely is an equilibrium to exist. The paper then turns to incomplete preferences with concave multi-utility representations. Several definitions of efficiency and of equilibrium with inertia are considered. Sufficient conditions and necessary and sufficient conditions are given for existence of efficient allocations and equilibria with inertia.

Keywords: Uncertainty, risk, risk adjusted prior, no arbitrage, equilibrium with short-selling, incomplete preferences, equilibrium with inertia.

JEL Classification: C62, D50, D81,D84,G1.

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1 Introduction

This paper reconsiders the theory of existence of efficient allocations and equilibria when consumption sets are unbounded below under the assumption that agents have incomplete preferences with concave multi-utility representations.

Following a long tradition and more particularly Hart [5] and Page [9], this work is motivated by an example in the theory of assets with short-selling. To simplify the analysis, markets are complete and a standard Arrow-Debreu model of exchange of contingent claims with short selling is considered. Agents are assumed not to have enough information to quantify uncertainty by a single probability, hence each agent has a set of priors. Agents are further assumed to have risk averse Bewley’s [1] incomplete preferences modeled by a unanimity rule (a contingent claim is better than another if its expected value under any prior is higher). As Bewley [1] suggested, an inertia principle is assumed in markets: agents do not trade to a contingent claim whose expected utility is not strictly higher under every prior than their status quo. Under standard conditions on utility indices (strict concavity and increasingness) and sets of priors (convexity and compactness), it is shown that a necessary and sufficient for existence of an individually rational efficient allocation or of an equilibrium is that the relative interiors of the sets of risk adjusted probabilities intersect. For this condition to be true, agents must have sets of priors with overlapping supports (where by support, we mean states of the world with positive probability for some prior) and their priors should not be too different. This condition generalizes the overlapping expectations given by Hart [5] for the case of single beliefs. Hart [5] further shows that the more risk averse the agents, the more likely is an equilibrium to exist. In pour model, we show that the more risk averse, the more ambiguity averse the agents, the more likely is an equilibrium to exist. Note that the condition that the relative interiors of the sets of risk adjusted probabilities is also equivalent to existence of equilibrium if agents have Gilboa-Schmeidler preferences with same sets of priors and utility indices. This may seem at odds with the literature on efficiency or equilibria with Bewley’s incomplete preferences and inertia (see Bewley’s [1], Rigotti and Shannon [10] or Dana and Riedel [3]) which highlights the differences with using maxmin complete preferences. However it is not: this paper discusses existence issues with shortselling and provides a condition which depends on the primitives of the model. It does not discuss whether a given allocation is efficient or whether there is no trade. This requires that the intersection of the sets of risk adjusted probabilities at the allocation (or at initial endowments) intersect.

The paper then turns to incomplete preferences with concave multi-utility representations which generalize Bewley’s [1] and Rigotti and Shannon [10] am-
bigness models with incomplete preferences. Several definitions of efficiency and equilibrium with inertia are considered. Sufficient conditions are given for existence of efficient allocations and equilibrium with inertia as well as necessary and sufficient conditions.

The paper is organized as follows: section 2 deals with the example and provides an existence theorem while section 3 deals with its generalization.

2 An example

2.1 Bewley preferences

We consider a standard Arrow-Debreu model of complete contingent security markets with short selling. There are two dates, 0 and 1. At date 0, there is uncertainty about which state \( s \) from a state space \( \Omega = \{1, \ldots, k\} \) will occur at date 1. At date 0, agents who are uncertain about their future endowments trade contingent claims for date 1. The space of contingent claims is the set of random variables from \( \Omega \to \mathbb{R} \). The random variable \( X \) which equals \( x_1 \) in state 1, \( x_2 \) in state 2 and \( x_k \) in state \( k \), is identified with the vector in \( \mathbb{R}^k \), \( X = (x_1, \ldots, x_k) \). Let \( \Delta = \{ \pi \in \mathbb{R}^k_+ : \sum_{s=1}^k \pi_s = 1 \} \) be the probability simplex in \( \mathbb{R}^k \). Let \( \text{int}\Delta = \{ \pi \in \Delta, \pi_s > 0, \forall s \} \). For \( A \subseteq \Delta \), \( \text{int}A = \{ \pi \in \Delta | \exists \text{ a ball } B(p, \varepsilon) \text{ s.t. } B(p, \varepsilon) \cap \text{int} \Delta \subseteq A \} \). For a given \( \pi \in \Delta \), we denote by \( E_{\pi}(X) := \sum_{l=1}^k \pi_l x_l \) the expectation of \( X \). Finally, for a given price \( p \in \mathbb{R}^k \), \( p \cdot X := \sum_{l=1}^k p_l x_l \), the price of \( X \).

There are \( m \) agents indexed by \( i = 1, \ldots, m \). Agent \( i \) has an endowment \( E^i \in \mathbb{R}^k \) of contingent claims. Let \( (E^i)_{i=1}^m \) be the \( m \)-tuple of endowments and \( E = \sum_{i=1}^m E^i \) be the aggregate endowment. We assume that agent \( i \) has a convex compact set of priors \( P^i \subseteq \text{int}\Delta \) and an incomplete Bewley preference relation \( \succeq^i \) over \( \mathbb{R}^k \) defined by

\[
X \succeq^i Y \iff E_{\pi}(u^i(X)) \geq E_{\pi}(u^i(Y)), \forall \pi \in P^i
\]  

where \( u^i : \mathbb{R} \to \mathbb{R} \) is a strictly concave, increasing differentiable utility index fulfilling \( u^i(0) = 0 \). The associated strict preference is \( X \succ^i Y \) if \( X \succeq^i Y \) and \( E_{\pi}(u^i(X)) > E_{\pi}(u^i(Y)) \) for some \( \pi \in P^i \).

2.2 Individual and collective absence of arbitrage

In this subsection, we define and characterize the useful trading directions of a Bewley preference relation of type (1). Our main result is that they coincide with those of a Gilboa-Schmeidler utility defined by \( (u, P) \) (see (2) below) and with those of a utility in the class of variational preferences also defined.
by \((u, P)\), introduced by Dana and Riedel [3], axiomatized by Mihm [7] and called reference-dependent ambiguity averse (RAA) utility. We then recall the concepts of no-arbitrage prices and of collective absence of arbitrage. As these concepts only depend on useful vectors, they also coincide.

### 2.2.1 Useful vectors

To simplify notations, in this subsection, the agent’s index is omitted. We consider an agent described by a pair \((u, P)\) of a utility index and a set of priors. For \(\pi \in P\), let

\[
\hat{P}_\pi(X) = \{ Y \in \mathbb{R}^k \mid E_\pi(u(Y)) \geq E_\pi(u(X)) \}
\]

be the set of contingent claims preferred to \(X\) for the utility \(E_\pi(u(.))\) and

\[
R_\pi(X) = \{ W \in \mathbb{R}^k \mid E_\pi(u(X + \lambda W)) \geq u(X), \ \forall \lambda \in \mathbb{R} \}
\]

be its asymptotic cone. As \(E_\pi(u(.))\) is concave, \(R_\pi(X)\) is independent of \(X\) and denoted \(R_\pi\). Taking \(X = 0\), we obtain:

\[
R_\pi = \{ W \in \mathbb{R}^k \mid E_\pi(u(\lambda W)) \geq u(0) = 0, \ \forall \lambda \in \mathbb{R} \}
\]

Let

\[
\hat{P}(X) = \{ Y \in \mathbb{R}^k \mid E_\pi(u(Y)) \geq E_\pi(u(X)), \ \forall \pi \in P \}
\]

be the set of contingent claims preferred to \(X\) for the Bewley’s preference defined by (1) and \(R(X)\) be its asymptotic cone. From Rockafellar’s [11] corollary 8.3.3, 
\[R(X) = \bigcap_{\pi \in P} R_\pi(X) = \bigcap_{\pi \in P} R_\pi.\]
Hence it is independent of \(X\) and denoted \(R\).

\[
R = \{ W \in \mathbb{R}^k \mid E_\pi(u(\lambda W)) \geq 0, \ \forall \lambda \in \mathbb{R}_+, \ \pi \in P \}
\]

and is called the set of useful vectors for \(\succeq\). For a given pair \((u, P)\), let

\[
V(X) = \min_{\pi \in P} E_\pi(u(X)) \quad (2)
\]

be the Gilboa-Schmeidler’s utility. As \(V(0) = 0\), we recall that the set of useful vectors for \(V\) is

\[
\{ W \in \mathbb{R}^k \mid V(\lambda W) \geq 0, \ \forall \lambda \in \mathbb{R} \} = \{ W \in \mathbb{R}^k \mid E_\pi(u(\lambda W)) \geq 0, \ \forall \lambda \in \mathbb{R}, \pi \in P \}
\]

Hence it coincides with \(R\). Furthermore given \(C \in \mathbb{R}^k\) a reference point, the \(C\)-reference dependent ambiguity averse (RAA) utility is defined by

\[
V_C(X) = \min_{\pi \in P} [E_\pi(u(X)) - E_\pi(u(C))] \quad (3)
\]

This is a concave variational utility, hence the set of useful vectors is independent of \(X\). As \(V_C(C) = 0\), \(R_{V_C} = \{ W \in \mathbb{R}^k \mid V_C(C + \lambda W) \geq 0, \ \forall \lambda \in \mathbb{R} \}\). Equivalently

\[
R_{V_C} = \{ W \in \mathbb{R}^k \mid E_\pi(u(C + \lambda W)) \geq E_\pi(u(C)), \ \forall \lambda \in \mathbb{R}, \pi \in P \} = R
\]

The previous discussion is summarized in the following lemma:
Lemma 1 The set of useful vectors for a Bewley preference of type (1) coincides with the set of useful vectors for a Gilboa-Schmeidler preference (2) or of an RAA utility (3) for any reference point \( C \in \mathbb{R}^k \).

Let us recall a characterization of useful vectors proven in Dana and Le Van [2]. To this end, let

\[
\tilde{P} = \left\{ p \in \Delta \mid \exists \pi \in P, Z \in \mathbb{R}^k \text{ s.t. } p_s = \frac{\pi_s u'(z_s)}{E_{\pi} u'(Z)}, \forall s = 1, \ldots, k \right\} \tag{4}
\]

be the set of risk adjusted probabilities. We have (see Dana and Le Van [2]) :

Lemma 2 \( R = \{ W \in \mathbb{R}^k \mid E_p(W) \geq 0, \text{ for all } p \in \tilde{P} \} \).

In the next two subsections, we introduce two concepts of absence of arbitrage, a concept of individual no-arbitrage and a concept of collective absence arbitrage. These concepts only depend on agents’ useful vectors.

2.2.2 No-arbitrage price

We first turn to the concept of no-arbitrage price.

Definition 1 A price vector \( p \in \mathbb{R}^k \) is a ”no-arbitrage price” for agent \( i \) if \( p \cdot W > 0 \), for all \( W \in \mathbb{R}^i \backslash \{0\} \). A price vector \( p \in \mathbb{R}^k \) is a ”no-arbitrage price” for the economy if it is a no-arbitrage price for each agent.

Let \( S^i \) denote the set of non arbitrage prices for \( i \). Using lemma 2, Dana and Le Van [2] characterize \( S^i \) in terms of risk adjusted probabilities.

Lemma 3 1. The set of no-arbitrage prices for agent \( i \) is \( S^i = \text{cone int } \tilde{P}^i \) where \( p \in \text{ int } \tilde{P}^i \) if and only if \( \exists \pi \in P^i \cap \text{ int } \Delta, Z \in \mathbb{R}^k, \forall s, a < u'(z_s) < b \) and \( p_s = \frac{\pi_s u'(z_s)}{E_{\pi} u'(Z)} \). The more risk averse, the more ambiguity averse the agent, the larger is \( S^i \).

2. The set of no-arbitrage prices for the economy is \( \cap S^i = \text{cone } \cap \text{ int } \tilde{P}^i \). The more risk averse, the more ambiguity averse the agents, the larger is the set of no-arbitrage prices for the economy.

2.2.3 Collective absence of arbitrage

From now on, a feasible trade is an \( m \)-tuple \( W^1, \ldots, W^m \) with \( W^i \in \mathbb{R}^k \) for all \( i \) and \( \sum_i W^i = 0 \). We recall the no-unbounded-arbitrage (NUBA) condition introduced by Page [8] which requires inexistence of unbounded utility nondecreasing feasible trades.

Definition 2 The economy satisfies the NUBA condition if \( \sum_i W^i = 0 \) and \( W^i \in R^i \) for all \( i \), implies \( W^i = 0 \) for all \( i \).
From Lemma 2, we may characterize the NUBA condition.

**Corollary 1** NUBA is equivalent to: there exists no feasible trade $W^1, \ldots, W^m$ with $W^i \neq 0$ for some $i$ that fulfills $E_\pi(W^i) \geq 0$ for all $i$ and $\pi \in P^i$.

### 2.3 Existence of efficient allocations and equilibria

#### 2.3.1 Concepts in equilibrium theory

We next recall standard concepts in equilibrium theory.

Given the allocation of initial endowments $(E^i)_{i=1}^m$, an allocation $(X^i)_{i=1}^m \in \mathbb{R}^k_m$ is attainable (or feasible) if $\sum_{i=1}^m X^i = E$. The set of individually rational attainable allocations $A((E^i)_{i=1}^m)$ is defined by

$$A((E^i)_{i=1}^m) = \left\{ (X^i)_{i=1}^m \in (\mathbb{R}^k)^m \mid \sum_{i=1}^m X^i = E \text{ and } X^i \geq E^i, \forall i \right\}.$$ \[\text{Definition 3}\]

Given $(E^i)_{i=1}^m$, an attainable allocation $(X^i)_{i=1}^m$ is $B$-efficient if there does not exist $(X'^i)_{i=1}^m$ attainable such that $X'^i \succeq X^i$ for all $i$ with a strict inequality for some $i$. It is weakly $B$-efficient if there does not exist $(X'^i)_{i=1}^m$ attainable such that $E_\pi(u^i(X'^i)) > E_\pi(u^i(X^i))$, $\forall \pi \in P^i$, $\forall i$. It is individually rational (weakly) $B$-efficient if it is (weakly) $B$-efficient and $X^i \geq E^i$ for all $i$.

Since $u^i$ is strictly concave and strictly increasing and $P^i$ is compact for all $i$, $(X^i)_{i=1}^m$ is $B$-efficient if and only if $(X^i)_{i=1}^m$ is weakly $B$-efficient (see Lemma 4 and Remark 3 below).

**Definition 4** A pair $(X^*, p^*) \in A((E^i)_{i=1}^m) \times \mathbb{R}^k \setminus \{0\}$ is a (weak) $B$-equilibrium with inertia if

1. for each agent $i$ and $X^i \in \mathbb{R}^k$, $X^i \succ X^i*$ $\left( E_\pi(u^i(X^i)) > E_\pi(u^i(X^i*)) \right)$, $\forall \pi \in P^i$, $\forall i$ implies $p^* \cdot X^i > p^* \cdot X^i*$,

2. for each agent $i$, $p^* \cdot X^i* = p^* \cdot E^i$.

Since $u^i$ is strictly concave for all $i$, it may easily be verified that $(X^*, p^*)$ is a $B$-equilibrium with inertia if and only if it is a weak $B$-equilibrium with inertia. Note that at an equilibrium with inertia, the allocation is individually rational.

#### 2.3.2 Existence of efficient allocations and equilibria

The following theorem fully characterizes existence of individually rational efficient allocations as well as equilibria with inertia.

**Theorem 1** The following assertions are equivalent:

1. there exists a no arbitrage price, equivalently $\cap_i \operatorname{int} \tilde{P}^i \neq \emptyset$,
2. there exists no feasible trade $W^1, \ldots, W^m$ with $W^i \neq 0$ for some $i$ and $E_\pi(W^i) \geq 0$ for all $\pi \in \tilde{P}^i$ and for all $i$,

3. the set of individually rational attainable allocations for the economy with Bewley preferences is compact

4. there exists an individually rational efficient allocation for the economy with Bewley’s preferences

5. there exists an equilibrium with inertia for the economy with Bewley’s preferences.

Furthermore any equilibrium price is a no-arbitrage price.

Proof: Since $X^i \succeq E^i$ is equivalent to $V_{E^i}(X) \geq V_{E^i}(E^i) = 0$, the set of individually rational attainable allocations for the economy with Bewley’s preferences coincides with the set of individually rational attainable allocations for the economy with RAA utilities with reference points $(E^i)_{i=1}^m$. Applying Dana and Le Van [2] to these utilities, we obtain the equivalence between 1, 2 and 3.

Let us prove that 1 implies 4. Applying Dana and Le Van [2] to RAA utilities with reference points $(E^i)_{i=1}^m$, we obtain the existence of an efficient allocation $(\bar{X}^i)_{i=1}^m$ for RAA utilities. Let us show that it is Bewley weakly efficient. Suppose not. Then there exists $(\bar{X}^i)_{i=1}^m$ attainable such that $E_\pi(u^i(\bar{X}^i)) > E_\pi(u^i(\bar{X}^i))$ for all $\pi \in P^i$ for all $i$ but then $V_{E^i}(X^i) > V_{E^i}(\bar{X}^i)$ for all $i$ contradicting the efficiency of $(\bar{X}^i)_{i=1}^m$ for RAA utilities. Therefore $(\bar{X}^i)_{i=1}^m$ is an individually rational Bewley weakly efficient allocation, hence efficient allocation. Let us next show that 4 implies 2. Let $(\bar{X}^i)_{i=1}^m$ be a Bewley efficient allocation. Suppose that there exists a feasible trade $W^1, \ldots, W^m$ with $W^i \in R^i$ for all $i$ and $W^i \neq 0$ for some $i$. We then have $E_\pi(u^i(\bar{X}^i + tW^i)) \geq E_\pi(u^i(\bar{X}^i))$ for all $\pi \in P^i$ for all $i$ and as $u^i$ is strictly concave, $E_\pi(u^i(\bar{X}^i + tW^i)) > E_\pi(u^i(\bar{X}^i))$ for all $\pi \in P^i$ for any $i$ such that $W^i \neq 0$. The allocation $(\bar{X}^i + tW^i)_{i \in I}$ being feasible, this contradicts the Bewley-efficiency of $(\bar{X}^i)_{i=1}^m$. Hence 1-4 are equivalent. Finally let us show that 1 is equivalent to 5. Let us first remark that if $(X^*, p^*)$ is an equilibrium with inertia, then $p^*$ is a no-arbitrage price. Indeed as $u^i$ is strictly concave, if $W^i$ is useful and $W^i \neq 0$, then $E_\pi(u^i(X^{si} + tW^i)) > E_\pi(u^i(X^{si})), \forall \pi \in P^i$, hence $p^* \cdot W^i > 0$. Hence if there exists an equilibrium, there exists a no-arbitrage price. Therefore 5 implies 1. Conversely if 1 holds true, then from Dana and Le Van [2], the economy with variational utilities $(V_{E^i})_{i \in I}$ has an equilibrium $(X^*, p^*)$. Let us show that $(X^*, p^*)$ is a weak Bewley equilibrium with inertia. Indeed if $E_\pi(u^i(X^i)) > E_\pi(u^i(X^{si})), \forall \pi \in P^i$, then $V_{E^i}(X) > V_{E^i}(X^{si})$, hence $p^* \cdot X^i > p^* \cdot X^{si}$. Furthermore as the family of utilities $(V_{E^i})_{i \in I}$ is strictly concave, if $X^{si} \neq E^i$, then $V_{E^i}(X^{si}) > V_{E^i}(E^i)$, hence $X^{si} > E^i$ proving that
\((X^*, p^*)\) is a weak Bewley equilibrium with inertia, hence a Bewley equilibrium with inertia.

**Remark 1** When stating theorem 1, we have assumed that all agents had Bewley’s incomplete preferences. From Dana and Le Van [2], we could have assumed that all agents had Gilboa-Schmeidler’s utilities. A more general result is true: agent \(i\) may either have a Bewley’s incomplete preference or a Gilboa-Schmeidler’s utility or an RAA utility with any reference point with utility index \(u^i\) and sets of priors \(P^i\) or any utility with useful vectors \(R^i\). Indeed if agent \(i\) has a Bewley’s incomplete preference, one can consider the fictitious agent with an RAA utility with reference point \(E^i\), utility index \(u^i\) and priors \(P^i\). From Dana and Le Van [2], we obtain the existence of an equilibrium for the fictitious economy. As in the proof of 1 implies 4, any equilibrium of the fictitious economy is an equilibrium of the original economy.

**Remark 2** The strict concavity of the utility functions plays an important and subtle role. First, for most purposes, when \(X \succ Y\), we may assume that \(E_\pi(u(X)) > E_\pi(u(Y)), \forall \pi \in \mathcal{P}\). Second, strict concavity is used to prove the equivalence between weak B-equilibrium and B-equilibrium and weak B-efficiency and B-efficiency. Third, as \(u\) is strictly concave, for any useful vector \(W \neq 0\) and any \(\pi \in \mathcal{P}\), the map \(t \rightarrow E_\pi(u(X + tW))\) is strictly increasing. In theorem 1, this is used in 4 implies 2 and in the assertion that an equilibrium price is a no-arbitrage price.

**Remark 3** In our model an efficient allocation exists iff \(\bigcap_{i} \text{int } \tilde{P}_i \neq \emptyset\) while a given attainable allocation \((X^i)_{i=1}^m\) is efficient iff the sets of risk-adjusted probabilities \(\tilde{P}_i(X^i)\) at \(X^i, \ i = 1, \ldots, m\) intersect (see Rigotti and Shannon [10]).

### 3 An abstract economy with incomplete preferences

#### 3.1 The economy

We consider an economy with \(m\) agents and \(d\) goods. Agents \(i\) has consumption space \(\mathbb{R}^d\), endowment \(E^i \in \mathbb{R}^d\) and incomplete (or complete) convex preferences over \(\mathbb{R}^d\), defined by a family \(\mathcal{U}^i: \mathbb{R}^d \to \mathbb{R}\) of concave utility functions as follows: \(X^i \in \mathbb{R}^d\) is preferred to \(Y^i \in E\) by agent \(i\), denoted \(X^i \succeq Y^i\) if \(u_i(X^i) \geq u_i(Y^i)\) for every \(u_i \in \mathcal{U}^i\). The associated strict preference is \(X ^i \succ Y\) if \(X ^i \succeq Y\) and \(u_i(X^i) > u_i(Y^i)\) for some \(u_i \in \mathcal{U}^i\). Let \(E = \sum_{i=1}^m E^i\) be the aggregate endowment.
We assume that for every $i$, the utilities in $U^i$ are everywhere finite, concave, and there is a topology on $U^i$ which makes it compact and such that the evaluation map $u \in U^i \mapsto u(X)$ is continuous for every $X \in \mathbb{R}^d$. We assume that $u(0) = 0$ for all $u \in U^i$ and all $i$. Let us give two examples of such families. For notational simplicity, we drop the subscript $i$.

**Example 1**

Let $T$ be a compact subset of $\mathbb{R}^p$ which can be interpreted as a subset of parameters and $U = \{ u : \mathbb{R}^d \times \mathbb{R}^p \to \mathbb{R} \}$, $u$ continuous on $\mathbb{R}^d \times T$ and $u(., t)$ concave for every $t \in T$.

**Example 2**

$U$ is a set of concave functions $\mathbb{R}^d \to \mathbb{R}$, closed for the topology of uniform convergence on compact subsets and uniformly bounded on each ball of $\mathbb{R}^d$. $U$ is then $k$-Lipschitz on the ball of radius $R$ for some constant $k_R$ which depends on the radius. From Ascoli’s theorem, $U$ is then compact for the topology of uniform convergence on compact subsets.

An allocation $(X_i)_{i \in I} \in \mathbb{R}^{dm}$ is feasible if $\sum X_i = E$. The set of individually rational attainable allocations $A((E_i)_{i=1}^m)$ is defined as in 2.3

**3.2 Arbitrage concepts**

We briefly redefine for abstract incomplete preferences, the concepts which were defined in section 2.2

$\hat{P}^i(X) = \{ Y \in \mathbb{R}^k \mid u(Y) \geq u(X), \forall u \in U^i \} = \bigcap_{u \in U^i} \{ Y \in \mathbb{R}^k \mid u(Y) \geq u(X) \}$

be the preferred set at $X$ by $i$ and $R^i(X)$ be its asymptotic cone. As the utilities $u \in U^i$ are concave, $R^i(X)$ is independent of $X$ and denoted $R^i$. It is called the set of useful vectors for $\succeq^i$. From Rockafellar’s [11] corollary 8.3.3,

$$R^i = \{ W \in \mathbb{R}^d \mid u(\lambda W) \geq 0, \forall \lambda \in \mathbb{R}, u \in U^i \}. $$

As discussed in the previous section, for any $C \in \mathbb{R}^d$, $R^i$ is also the set of useful vectors for any complete preference represented by a utility of the form $V_{E^i}(X) = \min_{u \in U^i} [u(X) - u(E^i)]$. Note that this utility is well defined under our assumptions.

A price vector $p \in \mathbb{R}^d$ is a ”no-arbitrage price” for agent $i$ if $p \cdot W > 0$, for all $W \in R^i \setminus \{0\}$. Let $S^i$ denote the set of non arbitrage prices for $i$. Then $S^i = -\text{int}(R^i)^0$ (where $(R^i)^0 = \{ p \in \mathbb{R}^d \mid p \cdot X \leq 0, \forall X \in R^i \}$). A
price vector \( p \in \mathbb{R}^k \) is a "no-arbitrage price" for the economy if it is a no-arbitrage price for each agent. A price vector \( p \in \mathbb{R}^k \) is a no-arbitrage price for the economy if and only if \( p \in \cap_i S^i = -\cap_i \text{int}(R^i)^0 \).

An \( m \)-tuple \( (W^1, \ldots, W^m) \in \mathbb{R}^{dm} \) is a feasible trade if \( \sum_i W^i = 0 \).

A trade \( W \in \mathbb{R}^{d \setminus \{0\}} \) is a half-line for a utility \( u : \mathbb{R}^d \to \mathbb{R} \) if there exists \( X \in \mathbb{R}^d \) such that \( u(X + \lambda W) = u(X) \) for all \( \lambda \geq 0 \). As \( u \) is concave, when it has no half-line, then if \( W \neq 0 \), \( u(X + \lambda W) > u(X) \) for every \( X \) and \( \lambda > 0 \).

### 3.3 Efficiency and equilibrium concepts

**Definition 5**

1. A feasible allocation \( (X^i)_{i=1}^m \) is efficient if there does not exist \( (X'^i)_{i=1}^m \) feasible such that \( X'^i \succeq X^i \) for all \( i \) with a strict inequality for some \( i \). It is weakly efficient if there does not exist \( (X'^i)_{i=1}^m \) feasible such that \( u^i(X'^i) > u^i(X^i) \), \( \forall u_i \in \mathcal{U}^i \), \( \forall i \). It is individually rational (weakly) efficient if it is (weakly) efficient and \( X^i \succeq E^i \) for all \( i \).

2. A pair \( (X^*, p^*) \in A((E^i)_{i=1}^m) \times \mathbb{R}^{d \setminus \{0\}} \) is an (weak) equilibrium with inertia if

   (a) for any \( i \) and \( X^i \in \mathbb{R}^d \), \( X^i \succ X'^i \) \( (u^i(X^i) > u^i(X'^i) \) for every \( u^i \in \mathcal{U}^i \) implies \( p^* \cdot X^i > p^* \cdot X'^i \),

   (b) for any \( i \), \( p^* \cdot X^i = p^* \cdot E^i \).

A common increasing direction, is a trade \( e \in \mathbb{R}^d \) such that \( e \cdot p > 0 \) for every \( (X, u) \in \mathbb{R}^d \times \mathcal{U}^i \), \( p \in \partial u(X) \) and every \( i \). When \( e \) is a common increasing direction, \( u(X-te) < u(X), u(X+te) > u(X) \) for every \( t \geq 0 \), \( X \in \mathbb{R}^d \), \( u \in \mathcal{U}^i \) and \( i \).

**Lemma 4**

Let \( u \) be strictly concave for every \( u \in \mathcal{U}^i \) and \( i \).

1. A pair \( (X^*, p^*) \in A((E^i)_{i=1}^m) \times \mathbb{R}^{d \setminus \{0\}} \) is an equilibrium with inertia if and only if it is a weak equilibrium with inertia.

2. If agents have a common increasing direction, then an attainable allocation \( (X^i)_{i=1}^m \) is efficient if and only if it is weakly efficient.

**Proof:** The proof of the first assertion is omitted. To prove the second, clearly, if \( (X^i)_{i=1}^m \) is efficient, it is weakly efficient. Let us show that weak efficiency implies efficiency. Assume that \( (X^i)_{i=1}^m \) is a weakly efficient allocation and that it is not efficient. W.l.o.g. assume that there exists a feasible allocation \( (Y^1)_{i=1}^m \) such that \( Y^1 \succ X^1 \) and \( Y^i \succeq X^i \), \( i \neq 1 \). By considering \( \frac{Y^1+X^i}{2} \) instead of \( Y^1 \), we may assume that \( u(Y^1) > u(X^1) \), \( \forall u \in \mathcal{U}^1 \). Since the map \( X \to u(X) \)
is continuous for every \( X \in \mathbb{R}^d \), for any \( u \in U^1 \), there exists \( \varepsilon_u \) such that \( u(Y^1 - e\varepsilon) > u(X^1) \) for every \( \varepsilon \leq \varepsilon_u \). For a given \( \varepsilon > 0 \), let

\[ V_\varepsilon = \{ u \in U^1 | u(Y^1 - e\varepsilon) > u(X^1) \} \]

\( V_\varepsilon \) is open and from the previous argument \( \cup_\varepsilon V_\varepsilon = U^1 \). Since \( U^1 \) is compact, there exists a finite subcovering of \( u \in U^1 \) by \((V_{\varepsilon_i})\). Let \( \varepsilon = \min_i \varepsilon_i \) and \( \varepsilon' = \frac{\varepsilon}{m-1} \). We then have

\[ Y^1 - e\varepsilon > X^1 \text{ and } Y^i + e\varepsilon' > X^i, \ i \neq 1 \]

contradicting the weak efficiency of \( X \).

\[ \square \]

### 3.4 An existence result

**Theorem 2** The following assertions are equivalent:

1. there exists a no arbitrage price for the economy \((\cap \text{int} \ (R^i)) \neq \emptyset \),

2. NUBA: \( \sum_i W^i = 0 \) and \( W^i \in R^i \) for all \( i \) implies \( W^i = 0 \) for all \( i \),

3. the set of individually rational attainable allocations is compact.

   Any of the previous assertions implies any of the following assertions:

4. there exists an individually rational weakly efficient allocation,

5. there exists a weak equilibrium with inertia.

   Furthermore,

6. if \( u \) has no half-line for every \( u \in U^i \) and \( i \), then assertions 1-5 are equivalent and any weak equilibrium price is a no-arbitrage price.

7. If \( u \) is strictly concave for every \( u \in U^i \) and \( i \), then

   (a) 1-5 are equivalent to the existence of an equilibrium

   (b) If agents have a common increasing direction, then 1-5 are equivalent to the existence of an individually rational efficient allocation.

**Proof:** As the set of useful vectors of \( \succeq^i \) and \( V_{E_i} \) coincide and the set of individually rational attainable allocations for the economy with utilities \((V_{E^i})_i \), coincides with the set of individually rational attainable allocations for the economy with preferences \((\succeq^i)_{i \in I} \), the equivalence between 1, 2 and 3 follows from standard results on arbitrage with complete preferences. It is also standard that any of these assertions implies the existence of an individually rational efficient allocation \((\bar{X}^i)_{i=1}^m \) or of an equilibrium \((X^*, p^*) \) for the economy with
utilities \((V_\mathcal{E})_i\). By the same proofs as in theorem 1, \((\bar{X}^*)_{i=1}^m\) is an individually rational weakly efficient allocation for the economy with preferences \((\succeq^*)_i\) and \((X^*, p^*)\) is a weak equilibrium with inertia. Let \(u\) have no half-line for every \(u \in \mathcal{U}_i\), and let \((X^*, p^*)\) be a weak equilibrium with inertia. Then for any useful vector \(W^i \neq 0\), \(u(X^i + tW^i) > u(X^i), \forall u \in \mathcal{U}_i\), for any \(t > 0\). Hence \(p^* \cdot W^i > 0\) which proves that \(p^*\) is a no-arbitrage price for the economy and that 5 implies 1. To show that 4 implies 2, let \((\bar{X}^i)_{i=1}^m\) be an individually rational weakly efficient allocation and suppose that there exists a feasible trade \(W^1, \ldots, W^m\) with \(W^i \in \mathcal{R}^i\) for all \(i\) and \(W^i \neq 0\) for some \(i\). We have \(u(\bar{X}^i + tW^i) \geq u(\bar{X}^i), \forall u \in \mathcal{U}_i\), for all \(i\), and \(u(\bar{X}^i + tW^i) > u(\bar{X}^i), \forall u \in \mathcal{U}_i\), for any \(i\) such that \(W^i \neq 0\). The allocation \((\bar{X}^i + tW^i)_{i \in I}\) being feasible, this contradicts the weak efficiency of \((\bar{X}^i)_{i=1}^m\). If \(u\) is strictly concave, then \(u\) has no half-line. Hence assertion 6 is fulfilled. Furthermore, from lemma 4, any weak equilibrium with inertia is an equilibrium with inertia and if agents have a common increasing direction, any weakly efficient allocation is efficient. ■

References


